Mean first passage time for anomalous diffusion

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When the random force acting on a particle diffusing in an interval $[0,L]$ and subjected to a constant external force is a Gaussian white noise, the ''Brownian'' mean-squared displacement is described by the seminal relation $\langle x^2\rangle = 2Dt^\gamma$ with $\gamma=1$. However, for more complicated random forces the diffusion may be slower ($\gamma<1$, ''subdiffusion'') or faster ($\gamma>1$, ''superdiffusion'') than the ''normal'' diffusion. For both these cases we calculated the mean free passage time (MFPT)—the time needed to reach one of the traps at boundaries. The simple formulas for the different diffusive regimes are compared quantitatively for the simplest case of the absence of an external field and for an initial position in the middle of the interval. It turns out that the MFPT's for anomalous diffusion can be both larger or smaller than that for normal diffusion depending on the values of the length of the interval and the diffusion coefficient. Moreover, the MFPT can show nonmonotonic changes with the degree of departure from normal diffusion.

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I. INTRODUCTION

Noise is a very common feature in many fields of physics, chemistry, biology, and social science [1]. A one-dimensional stochastic process $x(t)$ can be described either by a stochastic differential equation of the Langevin type, or by the Fokker-Planck equation for the probability density function $P(x,t)$ to find a particle at x at time t. For the simplest case of a constant external force F and the diffusion coefficient D this equation has the form

$$\frac{\partial P}{\partial t} = -F\frac{\partial P}{\partial x} + D\frac{\partial^2 P}{\partial x^2} \tag{1}$$

subject to the proper boundary and initial conditions. For diffusion on an interval $[0,L]$ with absorbing boundaries the boundary conditions are

$$P(0,t) = P(L,t) = 0 \tag{2}$$

with the conventional initial condition

$$P(x,t=0) = \delta(x-x_0). \tag{3}$$

It is easy to verify that the solution of Eq. (1) subject to conditions (2) and (3) has the form

$$P(x,t) = \frac{2}{L}\exp\left[\frac{F(x-x_0)}{2D} - \frac{F^2 t}{4D}\right]\sum_{n=1}^{\infty}\sin\left(\frac{\pi n x}{L}\right) \times \sin\left(\frac{\pi n x_0}{L}\right)\exp\left(-\frac{\pi^2 n^2 D t}{L^2}\right). \tag{4}$$

All the properties of the stochastic process $x(t)$ can be described through the use of the probability density function (4). The hallmark of the Brownian motion described by Eq. (1) is the expression for the mean-squared displacement $\langle (x-x_0)^2\rangle = \int_0^L x^2 P(x,t)dx$, which is equal, for $F=0$, to

$$\langle (x-x_0)^2\rangle = 2Dt. \tag{5}$$

This defines the normal diffusion process driven by Gaussian white noise.

Another interesting area of exploration in the theory of random processes is extrema statistics, where the question is, ''How long does it take a random quantity to reach a given value?'' The latter is defined by the first passage time—the time at which a stochastic process first reaches some ''critical value.'' For our problem the so-called mean first passage time T (MFPT) means the time elapsing before reaching one of the absorbing boundaries $x=0$ or $x=L$. It can be easily shown [1] that T is expressed in terms of $P(x,t)$ as

$$T = \int_0^L dx \int_0^\infty dt P(x,t) = \frac{2\pi}{DL^2}\sum_{n=1}^{\infty}\frac{n[1-(-1)^n]\exp(-Fx_0/2D)\sin(n\pi x_0/L)}{(n^2\pi^2/L^2 + F^2/4D^2)^2}. \tag{6}$$

The explicit form (4) of $P(x,t)$ has been used in the last equality in Eq. (6).

In the absence of an external force, $F=0$, Eq. (6) reduces to

$$T = \frac{x_0(L-x_0)}{2D}. \tag{7}$$

All of the preceding refers to ''normal'' diffusion. However, considerable interest has been attached in recent years to ''anomalous'' diffusion, characterized by the violation of the Brownian law (5), which is replaced by the more general form

$$\langle (x-x_0)^2\rangle \sim Dt^\gamma \tag{8}$$

with $\gamma<1$ (subdiffusion) or $\gamma>1$ (enhanced diffusion or superdiffusion) describing diffusion processes slower or faster than ordinary Brownian diffusion.

It is interesting to note that non-Brownian behavior [$\gamma=3$ in Eq. (8)] was mentioned as early as 1926 by Richardson [2] in the study of turbulence in the atmosphere. In fact, his explanation of this phenomenon (''eddies of many sizes acting together'') is close to the spirit of the modern view on turbulence. Different aspects of anomalous diffusion and its applications have been discussed in the review articles [3–5]. Additional references can be found in a recent article [6], in which the authors describe many important results including some of their own.

We limit ourselves here only by the comment that in statistical language superdiffusion can be induced by anomalously long random walks, while subdiffusion can be associated with an anomalously long waiting time between successive walks. A physical example of the long walks is the so-called Levy walks [7], while as an example of the long waiting time one can mention [8] the motion near stability islands imbedded within a chaotic sea, where a fast diffusion particle sticks at such islands for a long time.

Since the number of applications of anomalous diffusion is increasing dramatically, it is of some interest to study the problem of the mean free passage time for this case. That problem is just the aim of the present article. In the two sections that follow we show how to modify the method of calculation of the MFPT described in Eqs. (1)–(6) for anomalous diffusion. The final section contains discussion and conclusions.

II. MFPT FOR SUBDIFFUSION

From the different frameworks for describing anomalous diffusion, we choose to use the fractional calculus [9,10] where the first order time derivative in Eq. (1) is replaced by the fractional derivative of order α with $0<\alpha<1$, i.e.,

$$\frac{\partial^\alpha P}{\partial t^\alpha} = -F_\alpha \frac{\partial P}{\partial x} + D_\alpha \frac{\partial^2 P}{\partial x^2}. \tag{9}$$

The fractional derivative in Eq. (9) is understood in terms of the Riemann-Liouville integral

$$\frac{\partial^\alpha P(x,t)}{\partial t^\alpha} \equiv \frac{1}{\Gamma(-\alpha)} \int_0^t \frac{P(x,\tau)}{(t-\tau)^{1+\alpha}} d\tau. \tag{10}$$

Equations (9) and (10) have been derived for the field-free case, $F=0$, by Balakrishnan [11] using a generalization of Brownian motion, and solved in terms of the so-called Fox function by Schneider and Wyss [12]. The full solution of these equation for free boundary condition was given recently [13].

We will use the technique of the Laplace transform

$$f(s) = L\{f(t)\} = \int_0^\infty f(t)\exp(-st)dt \tag{11}$$

which when applied to Eq. (10) gives

$$L\left\{\frac{\partial^\alpha P(x,t)}{\partial t^\alpha}\right\} = s^\alpha P(x,s) - s^{\alpha-1}P(x,t=0)$$
$$= s^\alpha P(x,s) - s^{\alpha-1}\delta(x-x_0) \tag{12}$$

where the initial conditions of Eq. (3) have been used.

The Laplace transform of Eq. (9) yields

$$D_\alpha \frac{\partial^2 P(x,s)}{\partial x^2} - F_\alpha \frac{\partial P(x,s)}{\partial x} = s^\alpha P(x,s) - s^{\alpha-1}\delta(x-x_0). \tag{13}$$

It is convenient to eliminate the term $\partial P/\partial x$ from Eq. (13) by defining a new function $W(x,t)$ as

$$P(x,s) = \exp\left(\frac{F_\alpha x}{2D_\alpha}\right) W(x,s), \tag{14}$$

which, finally, results in

$$D_\alpha \frac{\partial^2 W(x,s)}{\partial x^2} - \left(\frac{F_\alpha^2}{4D_\alpha} + s^\alpha\right) W(x,s)$$
$$= -\delta(x-x_0)s^{\alpha-1}\exp\left(-\frac{F_\alpha x}{2D_\alpha}\right). \tag{15}$$

Due to the singular term $\delta(x-x_0)$ in Eq. (15) one has to solve this equation separately in two intervals $[0,x_0]$ and $[x_0,L]$, and then match these solutions at $x=x_0$. Designating these solutions as W_1 and W_2, respectively, one obtains

$$W_1(x,s) = C_1\exp(\gamma x) + C_2\exp(-\gamma x),$$
$$W_2(x,s) = C_3\exp(\gamma x) + C_4\exp(-\gamma x), \tag{16}$$

where

$$\gamma = \sqrt{\frac{F_\alpha^2}{4D_\alpha^2} + \frac{s^\alpha}{D_\alpha}}. \tag{17}$$

The constants $C_1, \ldots, C_4$ have to be found from the boundary conditions $W_1(x=0,s) = W_2(x=L,s) = 0$ and from the matching conditions at $x=x_0$. The latter are the continuity of functions $W_1(x=x_0,s) = W_2(x=x_0,s)$, and the connection between the derivatives of these functions, which can be found by integrating Eq. (15) with respect to x between $x=x_0-\epsilon$ and $x=x_0=\epsilon$ with the infinitesimal ϵ. This yields

$$D_\alpha\left(\frac{dW_2}{dx}\right)_{x=x_0+\epsilon} - D_\alpha\left(\frac{dW_1}{dx}\right)_{x=x_0-\epsilon}$$
$$= -s^{\alpha-1}\exp\left(-\frac{F_\alpha x_0}{2D_\alpha}\right). \tag{18}$$

Solving these four equations for C_1–C_4 and substituting them into Eqs. (16) and (14) one gets

$$P_1(x,s) = \frac{\exp[F_\alpha(x-x_0)/2D_\alpha]\sinh[\gamma(L-x_0)]\sinh(\gamma x)}{D_\alpha \gamma s^{1-\alpha}\sinh(\gamma L)},$$

$$P_2(x,s) = \frac{\exp[F_\alpha(x-x_0)/2D_\alpha]\sinh[\gamma(L-x)]\sinh(\gamma x_0)}{D_\alpha \gamma s^{1-\alpha}\sinh(\gamma L)}. \tag{19}$$

In accordance with Eq. (6) the MFPT can be obtained from $P(x,t)$ by integration over x and t,

$$T=\int_0^\infty dt\int_0^L dxP(x,t)$$
$$=\int_0^\infty dt\frac{1}{2\pi i}\int_C ds\,\exp(st)\times\left[\int_0^{x_0}dxP_1(x,s)+\int_{x_0}^L dxP_2(x,s)\right]$$
$$=\int_0^\infty dt\frac{1}{2\pi i}\int_C ds\,\exp(st)\times\left[1-\frac{\sinh(\sqrt{s^\alpha/D_\alpha}(L-x_0))\exp(-F_\alpha x_0/2D_\alpha)}{\sinh\sqrt{s^\alpha/D_\alpha}L}-\frac{\sinh(\sqrt{s^\alpha/D\alpha}x_0)\exp[F_\alpha(L-x_0)/2D_\alpha]}{\sinh(\sqrt{s^\alpha/D_\alpha}L)}\right], \tag{20}$$

where Eq. (19) has been substituted in Eq. (20), and integration over x has been performed in the last equality in Eq. (20).The contour C in Eq. (20) and subsequent equations is the usual Bromich contour for the inverse Laplace transform.

Three integrals in Eq. (20) have a pole at the origin while the second and third integrals also have extra poles at the zeros of $\sinh(\sqrt{s^\alpha/D_\alpha}L)$, i.e., at $i\sqrt{s^\alpha/D_\alpha}L=n\pi$ or $s=-(n^2\pi^2D_\alpha/L^2)^{1/\alpha}$, $(n=1,2,3,\dots)$. The residues at the poles can be easily calculated, and one finds

$$\frac{1}{2\pi i}\int_C\frac{ds}{s}\frac{\sin(As^{\alpha/2})}{\sin(Bs^{\alpha/2})}=\left\{\frac{\sinh(AF_\alpha/2D_\alpha)}{\sinh(BF_\alpha/2D_\alpha)}+2\sum_{n=1}^\infty(-1)^n\times\exp\left[-\left(\frac{F_\alpha^2}{4D_\alpha}+\frac{n^2\pi^2}{B^2}\right)^{1/\alpha}t\right]\frac{\sin(n\pi A/B)}{\alpha n\pi A/B}\times\frac{\pi^2n^2D_\alpha/B^2}{(F^2/4D_\alpha+\pi^2n^2D_\alpha/B^2)}\right\}. \tag{21}$$

On substituting Eq. (21) into Eq. (20) one gets

$$T=1-\frac{\sinh[F_\alpha(L-x_0)/2D_\alpha]}{\sinh[F_\alpha L/2D_\alpha]}-\frac{\sinh[F_\alpha x_0/2D_\alpha]}{\sinh[F_\alpha L/2D_\alpha]}+\frac{2\pi\exp(-F_\alpha x_0/2D_\alpha)}{\alpha L^2D_\alpha^{1/\alpha}}\times\sum_{n=0}^\infty\frac{n[\exp(F_\alpha L/2D_\alpha)-(-1)^n]\sin(n\pi x_0/L)}{(n^2\pi^2/L^2+F_\alpha^2/4D_\alpha^2)^{1+1/\alpha}}. \tag{22}$$

For $F=0$, Eq. (19) reduces to

$$P_1(x,s)=\frac{\sinh[\sqrt{s^\alpha/D_\alpha}(L-x_0)]\sinh(\sqrt{s^\alpha/D_\alpha}x)}{\sqrt{D_\alpha}s^{1-\alpha/2}\sinh(\sqrt{s^\alpha/D_\alpha}L)};$$

$$P_2(x,s)=\frac{\sinh[\sqrt{s^\alpha/D_\alpha}(L-x)]\sinh(\sqrt{s^\alpha/D_\alpha}x_0)}{\sqrt{D_\alpha}s^{1-\alpha/2}\sinh(\sqrt{s^\alpha/D\alpha L})} \tag{23}$$

and Eq. (22) becomes

$$T=\frac{2}{\alpha\pi}\left(\frac{L}{\pi\sqrt{D_\alpha}}\right)^{2/\alpha}\sum_{n=1}^\infty\frac{[1-(-1)^n]\sin(n\pi x_0/L)}{n^{1+2/\alpha}}. \tag{24}$$

The sum over n in Eq. (24) contains only odd powers of n; therefore one finally obtains

$$T=\frac{4}{\alpha\pi}\left(\frac{L}{\pi\sqrt{D_\alpha}}\right)^{2/\alpha}\sum_{m=1}^\infty\frac{\sin[(2m+1)\pi x_0/L]}{(2m+1)^{1+2/\alpha}}. \tag{25}$$

Let us check that for normal diffusion, $\alpha=1$, Eq. (24) reduces to (6). To this end, taking the second derivative of Eq. (24) with respect to x_0 and using the well-known relations [14] $\Sigma_{k=0}^\infty\sin(kx)/k=(\pi-x)/2$ and $\Sigma_{k=0}^\infty(-1)^k\sin(kx)/k=x/2$, one obtains $d^2T/dx^2=-1/D$ with $D_{\alpha=1}=D$, which leads to Eq. (7).

Let us assume, for simplicity, that at $t=0$ a particle was placed in the middle of the interval $[0,L]$. Then, $x_0=L/2$, and $\sin(n\pi x_0/L)=\sin(n\pi/2)$ vanishes for even n and is equal to $(-1)^m$ for odd $n=2m+1$. Consequently, Eq. (25) reduces to

$$T=\frac{4}{\alpha\pi}\left(\frac{L}{\pi\sqrt{D_\alpha}}\right)^{2/\alpha}\sum_{n=1}^\infty\frac{(-1)^m}{(2m+1)^{1+2/\alpha}}. \tag{26}$$

Taking into account that [14] $\Sigma_{m=0}^\infty(-1)^m/(2m+1)^3=\pi^2/32$, one obtains the correct limit $T=L^2/8D$ for normal diffusion, $\alpha=1$, with $x_0=L/2$.

The sum in Eq. (26) resembles the Riemann zeta function, and can be expressed in integral form [15] so that Eq. (26) can be rewritten as

$$T=\frac{4}{\alpha\pi}\left(\frac{L}{\pi\sqrt{D_\alpha}}\right)^{2/\alpha}\frac{1}{2\Gamma(1+2/\alpha)}\int_0^\infty\frac{t^{2/\alpha}dt}{\cosh(t)}. \tag{27}$$

In the final section we compare the results obtained above for subdiffusion with those obtained in the next section for superdiffusion.

III. MFPT FOR SUPERDIFFUSION

Analogous to Eq. (9) one can write the Fokker-Planck equation for superdiffusion in the form

$$\frac{\partial P(x,t)}{\partial t}=-F_\beta\frac{\partial P(x,t)}{\partial x}+D_\beta\frac{\partial^\beta P(x,t)}{\partial x^\beta}, \tag{28}$$

where $0<\beta<2$. Using the Fourier transform with respect to the coordinate x,

$$P(q,t)=\int_{-\infty}^\infty dxP(x,t)\exp(-iqx),$$

$$P(x,t)=\int_{-\infty}^{\infty}dqP(q,t)\exp(iqx), \tag{29}$$

one can rewrite Eq. (28) as

$$\frac{\partial P(q,t)}{\partial t}+[D_\beta|q|^\beta+iqF_\beta]P(q,t)=0. \tag{30}$$

The solution of Eq. (30) has the form $P(q,t)=C\exp[(-D_\beta|q|^\beta-iqF_\beta)t]$ where the constant C has to be found from the initial condition. Assume that initially a particle was at $x=x_0$, i.e., $P(x,t=0)=\delta(x-x_0)$ or $P(q,t=0)=\exp(-iqx_0)$. Then the solution of Eq. (30) is

$$P(q,t)=\exp[-iqx_0-(D_\beta|q|^\beta+iqF_\beta)t]. \tag{31}$$

Equation (31) is the Laplace transform of the well-known symmetry Levy stable density [16] with $\langle x^m\rangle=\infty$ for $m>\beta$. The latter follows from the properties of the characteristic function $\langle x^n(t)\rangle=i^n[d^nP(q,t)/dq^n]|_{q=0}$. Superdiffusion with $1<\beta<2$ was considered recently by Barkai and Silbey [17] who discuss also the problem of diverging moments.

The boundary conditions for Eq. (28), $P(x=0,t)=P(x=L,t)=0$, can be written for the Fourier transforms $P(q,t)$ by Eq. (29) as

$$\int_{-\infty}^{\infty}dqP(q,t)=0 \tag{32}$$

and

$$\int_{-\infty}^{\infty}dqP(q,t)\exp(iqL)=0. \tag{33}$$

Condition (32) will be obeyed if $P(q,t)$ is an odd function of q, i.e., one has to take the imaginary part of Eq. (31), namely,

$$P(q,t)=\sin[q(x_0+F_\beta t)]\exp(-D_\beta|q|^\beta t). \tag{34}$$

Both the real and imaginary parts of the integral in Eq. (33) will vanish if and only if

$$q=\frac{n\pi}{L}. \tag{35}$$

On substituting Eq. (35) into Eqs. (34) and (29) and taking the sum over n in order to satisfy the initial condition, one finally obtains

$$P(x,t)=\frac{2}{L}\sum_{n=1}^{\infty}\sin\left[\frac{n\pi}{L}(x_0+F_\beta t)\right]\sin\left[\frac{n\pi}{L}(x+F_\beta t)\right]\times\exp\left[-D_\beta\left(\frac{n\pi}{L}\right)^\beta t\right]. \tag{36}$$

We are interested in the mean free passage time T for one-dimensional diffusion on a segment [0,L] which, according to Eq. (6) can be obtained from the distribution function $P(x,t)$ by the integration (36) over x and t, which gives

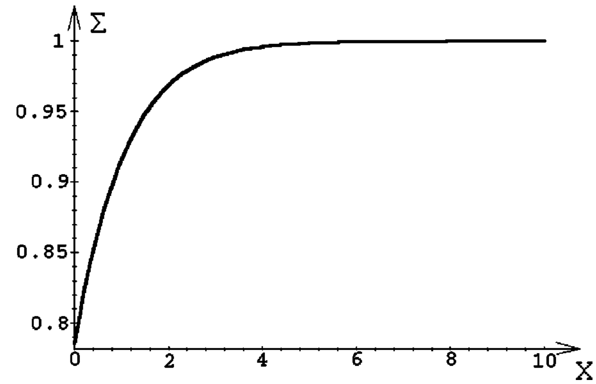

FIG. 1. The characteristic sum $\Sigma_{m=0}^{\infty}(-1)^m/(2m+1)^{1+x}$ appearing in the expressions (40) for MFPT as a function of x. Normal diffusion corresponds to $x=2$, and this sum is equal to $\pi^3/32$.

$$T=\frac{2}{\pi D_\beta}\left(\frac{L}{\pi}\right)^\beta\sum_{n=1}^{\infty}\frac{[1-(-1)^n]\sin(n\pi x_0/L)}{n^{1+\beta}}\times\frac{2(n\pi F_\beta/L)^2+D_\beta^2(n\pi/L)^{2\beta}}{4(n\pi F_\beta/L)^2+D_\beta^2(n\pi/L)^{2\beta}}. \tag{37}$$

For $F_\beta=0$ Eq. (37) reduces to

$$T=\frac{2}{\pi D_\beta}\left(\frac{L}{\pi}\right)^\beta\sum_{n=1}^{\infty}\frac{[1-(-1)^n]\sin(n\pi x_0/L)}{n^{1+\beta}}=\frac{4}{\pi D_\beta}\left(\frac{L}{\pi}\right)^\beta\sum_{n=1}^{\infty}\frac{(-1)^m\sin[(2m+1)\pi x_0/L]}{(2m+1)^{1+\beta}}. \tag{38}$$

If at $t=0$ a particle was placed in the middle of the interval $[0,L]$, i.e., $x_0=L/2$, then Eq. (38) reduces to

$$T=\frac{4}{\pi D_\beta}\left(\frac{L}{\pi}\right)^\beta\sum_{n=1}^{\infty}\frac{(-1)^m}{(2m+1)^{1+\beta}}, \tag{39}$$

which for normal diffusion, $\beta=2$, yields the correct result $T=L^2/8D$.

IV. DISCUSSION AND CONCLUSION

The mean first passage time of a diffusive particle moving in an interval $[0,L]$ with absorbing boundaries and subjected to a constant force is given by Eq. (6) which, in the absence of the force, reduces to the simple expression (7). This well-known result refers to ''normal'' diffusion when a white Gaussian noise is acting on a particle. This Brownian motion is characterized by the mean-squared displacement $\langle x^2\rangle=2Dt$.

However, in many cases the random force acting on a particle is more complicated than the simplest white noise. As a result, by the action of such a noise on a particle its mean-squared displacement may change in time more slowly (''subdiffusion'') or faster (''superdiffusion'') than that of a Brownian particle. For both these cases we found the mean free passage time T which is an important characteristic of the random process defining the average time needed for a

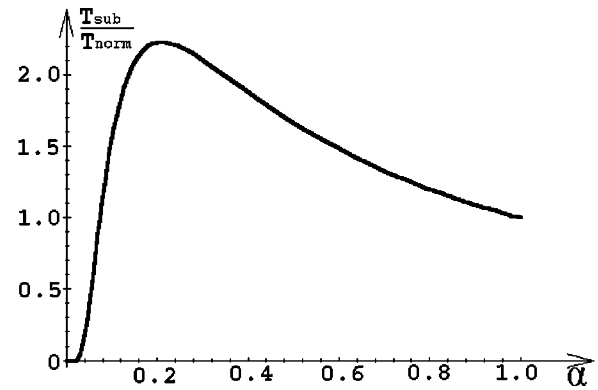

FIG. 2. The ratio of the MFPT of subdiffusion to that of normal diffusion, T_{sub}/T_{norm}, as a function of the ''parameter of subdiffusion'' α ($\alpha=1$ corresponds to normal diffusion) for $L^2/8D=1$.

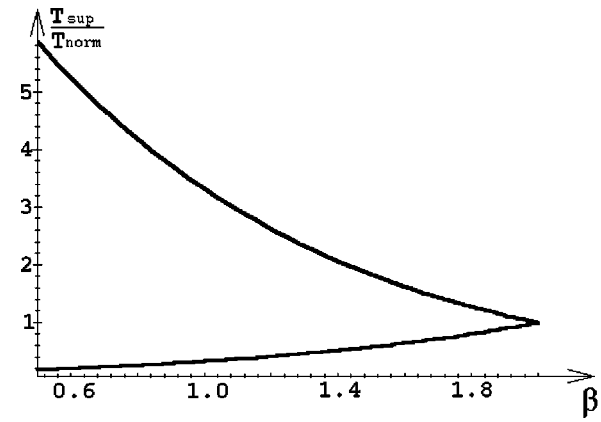

FIG. 3. The ratio of MFPT of superdiffusion to that of normal diffusion, T_{super}/T_{norm}, as the function of the ''parameter of superdiffusion'' β ($\beta=2$ corresponds to normal diffusion) for $L^2/8D=1$ and different values of the diffusion constant D_β. The upper graph is obtained for $D_\beta=0.1$, and the lower graph for $D_\beta=10$.

particle to be trapped by one of the boundaries of the interval $[0,L]$.

Solutions of the Fokker-Planck equation for anomalous diffusion can normally be found after passing to the Fourier-Laplace transforms. However, the inverse transformations are quite complicated, leading to some combinations of the Fox functions [6]. In particular, one has to mention some recent articles [6,18] which appeared after completion of our work. On the other hand, for calculation of the MFPT one can first perform integration over x, and then the inverse Fourier-Laplace transformation presents no problems.

The ''Brownian'' expression (7) for the MFPT is replaced by Eq. (22) for subdiffusion, and by Eq. (37) for superdiffusion. To decrease the number of parameters we compare the MFPT for these three cases in the simplest case of the absence of an external field and for an initial position of a particle in the middle of the interval. The appropriate formulas are Eqs. (7), (26), and (39), which, for convenience, we rewrite here in slightly different form

$$T_{norm}=\frac{4L^2}{\pi^3 D}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^3}=\frac{L^2}{8D}, \quad (40)$$

$$T_{sub}=\left(\frac{L^2}{8D_\alpha}\right)^{1/\alpha}\frac{4}{\alpha\pi}\left(\frac{8}{\pi^2}\right)^{1/\alpha}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^{1+2/\alpha}},$$

$$T_{\rm super}=\left(\frac{L^2}{8D_\beta}\right)^{\beta/2}\frac{4}{\pi D}\left(\frac{8D_\beta}{\pi^2}\right)^{\beta/2}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^{1+\beta}}.$$

Notice that the MFPT's have a similar functional form apart from the coefficients of different dimensions since the diffusion coefficients, according to Eqs. (9) and (28), have the dimensions $(\text{length})^2/(\text{time})^\alpha$ and $(\text{length})^\beta/(\text{time})$ for the sub- and superdiffusion, respectively. Of course, the two last expressions in Eq. (40) reduce to the case of normal diffusion when $\alpha=1$ and $\beta=2$.

In Figs. 1–3 we compare the MFPT for different diffusive regimes. All of them contain the sum $\Sigma_{m=0}^{\infty}(-1)^m/(2m+1)^{1+x}$, which we show in Fig. 1 as a function of x. This sum is equal to $\pi^3/32$ for normal diffusion ($x=2$), while the parameters $\alpha=2/x$ and $B=x$ correspond to subdiffusion and superdiffusion, respectively. The ratio of the MFPT for subdiffusion to that of normal diffusion is shown in Fig. 2 as a function of the parameter α, $0<\alpha\leqslant 1$, for $L^2/8D=1$. As one can see from Fig. 2 the ratio T_{sub}/T_{norm} is a nonmonotonic function of α. However, the ratio T_{sub}/T_{norm} will stay monotonic for different values of the parameter $L^2/8D_\alpha$, remaining larger (for $L^2/8D_\alpha>1$) or smaller (for $L^2/8D_\alpha<1$) than the MFPT for normal diffusion.

In contrast to subdiffusion, the ratio of the MFPT for superdiffusion to that for normal diffusion depends on the parameters L and D_β both in the ratio $L^2/8D_\beta$ and individually. In Fig. 3 we show the ratio T_{super}/T_{norm} as a function of the parameter β for $L^2/8D=1$ and different values of D_β. As one can see from this figure, the MFPT for superdiffusion can approach its value for normal diffusion at $\beta=2$, remaining larger (for $D_\beta=0.1$) or smaller (for $D_\beta=10$) than the MFPT for normal diffusion.

In conclusion, we found the mean free passage time for both subdiffusion and superdiffusion of a particle moving in the interval $[0,L]$, which turn out to be smaller or larger than the mean free passage time for normal diffusion depending on the parameters of the system.

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